Non-Geometric T-duality from Higher Groupoid Bundles with Connections



Christian Saemann Maxwell Institute and School of Mathematical and Computer Sciences Heriot–Watt University, Edinburgh

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Topological T-duality

- $\bullet\,$ String theories on backgrounds with U(1)-isometries:
 - \Rightarrow a T-dual partner
- Low-energy limit: corresponding supergravity contains *B*-field:
 ⇒ connective structure on a gerbe

Geometric string background:

- A Riemannian manifold X
- A principal/affine torus bundle $\pi: P \to X$
- $\bullet\,$ An abelian gerbe ${\mathscr G}$ on the total space of P

Topological T-duality

Geometric string background:

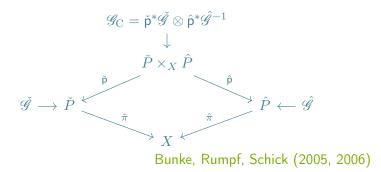
- A Riemannian manifold X
- A principal/affine torus bundle $\pi: P \to X$
- An abelian gerbe \mathscr{G} on the total space of P

Topological T-duality follows from exactness of the Gysin sequence: $\dots \to \mathrm{H}^{3}(X,\mathbb{Z}) \xrightarrow{\pi^{*}} \mathrm{H}^{3}(P,\mathbb{Z}) \xrightarrow{\pi_{*}} \mathrm{H}^{2}(X,\mathbb{Z}) \xrightarrow{F \cup} \mathrm{H}^{4}(X,\mathbb{Z}) \to \dots$

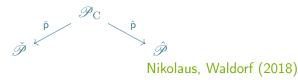
- Gerbe over P: 3-form $H \in \mathrm{H}^3(P,\mathbb{Z})$
- Fiber integration $\pi_*H = \hat{F} \in \mathrm{H}^2(X,\mathbb{Z})$ with $F \cup \hat{F} = 0$
- \Rightarrow There is $\hat{H} \in \mathrm{H}^3(P,\mathbb{Z})$ with $\pi_*\hat{H} = F$.
- Topological T-duality: $(F, H) \leftrightarrow (\hat{F}, \hat{H})$. Note: possibility for topology change!

Bouwknegt, Evslin, Hannabuss, Mathai (2004)

Topological T-duality, geometrically T-correspondence:



Principal 2-bundles (without connections):



Two open problems

- I. T-duality can lead to non-geometric backgrounds:
 - F^3 : *H* has no legs along fiber

T-duality: identity

 F^2 : H has 1 leg along fiber

 $\mathsf{T}\text{-duality} \to \mathsf{geometric\ string\ background}$

 F^1 : *H* has 2 legs along fiber

T-duality \rightarrow Q-space, (e.g. T-folds) locally geometric

 F^0 : H has all legs along fiber

T-duality \rightarrow *R*-space, non-geometric

Nikolaus/Waldorf cover $F^2 \leftrightarrow F^2$ and $F^2 \leftrightarrow F^1$ T-dualities What about the general case?

II. Differential refinement of this picture

Why is this interesting/hard?

- I. need to use suitable groupoids and augmented groupoids
- II. connections on principal 2-bundles require adjustment

Outline

- Connections on principal 2-bundles
- T-duality with differentially refined principal 2-bundles
- Explicit example: Nilmanifolds
- The T-duality group from Kaluza-Klein reduction
- Groupoid bundles for T-folds
- Augmented groupoid bundles for R-spaces

Principal 2-bundles or Non-Abelian Gerbes

with Connections

Christian Saemann Perturbative QFT, CK-duality, and Homotopy Algebras

Categorification

A mathematical structure ("Bourbaki-style") consists of

• Sets • Structure Functions • Structure Equations "Categorification":

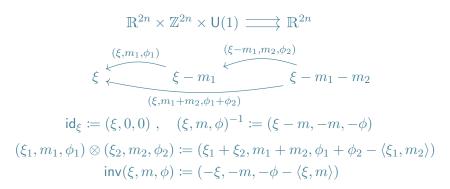
 $\label{eq:Sets} \begin{array}{l} \mathsf{Sets} \to \mathsf{Categories} \\ \mathsf{Structure} \ \mathsf{Functions} \to \mathsf{Structure} \ \mathsf{Functors} \\ \mathsf{Structure} \ \mathsf{Equations} \to \mathsf{Structure} \ \mathsf{Isomorphisms} \end{array}$

Example: Group \rightarrow 2-Group

- Set $G \rightarrow Category \mathscr{G}$
- $\bullet\,$ product, identity (1 : $* \to \mathsf{G}),$ inverse \to Functors
- $a(bc) = (ab)c \rightarrow Associator a : a \otimes (b \otimes c) \Rightarrow (a \otimes b) \otimes c$
- $\mathbb{1}a = a\mathbb{1} = a \to \mathsf{Unitors} \ \mathsf{I}_a : a \otimes \mathbb{1} \Rightarrow a, \ \mathsf{r}_a : \mathbb{1} \otimes a \Rightarrow a$
- $aa^{-1} = a^{-1}a = 1 \rightarrow \text{weak inv. inv}(x) \otimes x \Rightarrow 1 \leftarrow x \otimes \text{inv}(x)$

Note: Process not unique, variants: weak/strict/...

Example: The Lie 2-group $\underline{\mathsf{TD}}_n$



This Lie 2-group corresponds to a crossed module of Lie groups:

$$\begin{aligned} \mathsf{TD}_n &:= \left(\mathbb{Z}^{2n} \times \mathsf{U}(1) \stackrel{\mathsf{t}}{\longrightarrow} \mathbb{R}^{2n} \right) \\ \mathsf{t}(m, \phi) &:= m \\ \xi \triangleright (m, \phi) &:= (m, \phi - \langle \xi, m \rangle) \end{aligned}$$

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Principal 2-Bundles

Categorify bundles over a manifold M subordinate to cover (U_a) Breen, Messing (2005), Aschieri, Cantini, Jurčo (2005) Principal (H \xrightarrow{t} G)-bundle Object Principal G-bundle Cochains (q_{ab}) valued in G (q_{ab}) valued in G, (h_{abc}) valued in H Cocycle $g_{ab}g_{bc} = g_{ac}$ $t(h_{abc})g_{ab}g_{bc} = g_{ac}$ $h_{acd}h_{abc} = h_{abd}(q_{ab} \triangleright h_{bcd})$ Coboundary $g_a g'_{ab} = g_{ab} g_b$ $g_a g'_{ab} = t(h_{ab})g_{ab}g_b$ $h_{ac}h_{abc} = (g_a \triangleright h'_{abc})h_{ab}(g_{ab} \triangleright h_{bc})$ $A_a \in \Omega^1(U_a) \otimes \mathfrak{g}, B_a \in \Omega^2(U_a) \otimes \mathfrak{h}$ gauge pot. $A_a \in \Omega^1(U_a) \otimes \mathfrak{g}$ $\mathcal{F}_a = \mathrm{d}A_a + \frac{1}{2}[A_a, A_a] - \mathsf{t}(B_a) \stackrel{!}{=} 0$ Curvature $F_a = dA_a + A_a \wedge A_a H_a = \mathrm{d}B_a + A_a \triangleright B_a$ Gauge trafos $\tilde{A}_a := q_a^{-1} A_a q_a + q_a^{-1} dq_a$ $\tilde{A}_a := q_a^{-1} A_a q_a + q_a^{-1} \mathrm{d} q_a + \mathrm{t}(\Lambda_a)$ $\tilde{B}_a := q_a^{-1} \triangleright B_a + \tilde{A}_a \triangleright \Lambda_a + d\Lambda_a - \Lambda_a \wedge \Lambda_a$

Remarks:

- A principal $(1 \xrightarrow{t} G)$ -bundle is a principal G-bundle.
- A principal $(U(1) \xrightarrow{t} 1) = BU(1)$ -bundle is an abelian gerbe.

Why should the fake curvature(s) vanish?

$$\mathcal{F} := \mathrm{d}A + \frac{1}{2}[A, A] + \mathsf{t}(B) \stackrel{!}{=} 0$$

Without this condition:

- Gauge transformations do not close
- Cocycles do not glue together
- Higher parallel transport is not reparametrization invariant
- 6d Self-duality equation $H = \star H$ is not gauge-covariant:

 $H \to \tilde{H} = g \vartriangleright H - \mathcal{F} \rhd \Lambda$

With this condition:

- Principal $(1 \xrightarrow{t} G)$ -bundle is flat principal G-bundle.
- Higher connections are locally abelian!

Gastel (2019), CS, Schmidt (2020)

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Solution: Adjustment

Many (not all!) higher gauge groups come with

Adjustment of higher group \mathcal{G} :

CS, Schmidt (2020), Rist, CS, Wolf (2022)

- Additional map $\kappa: \mathcal{G} \times \text{Lie}(\mathcal{G}) \rightarrow \text{Lie}(\mathcal{G})$
- Necessary for consistent definition of invariant polynomials.
- From Alternator (\Rightarrow EL_{∞} -algebras, Borsten, Kim, CS (2021))

For connections on principal *G*-bundles:

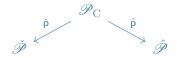
- Adjustment of curvature/cocycle/coboundary relations
- Can drop fake flatness condition

Example: Heterotic supergravity

- Lie 2-algebra $\mathfrak{g} imes \mathbb{R}
 ightrightarrow \mathfrak{g}$ or L_∞ -algebra $\mathbb{R}
 ightarrow \mathfrak{g}$
- $H = dB \frac{1}{3!}(A, [A, A]) (A, F) = dB + cs(A)$
- such that F arbitrary and dH = (F, F) follows

Geometric T-duality

Geometric T-duality: General Picture



- Nikolaus/Waldorf: Topological part:
 - $\check{\mathscr{P}}$ and $\hat{\mathscr{P}}$ are principal $\mathsf{TB}_n^{\mathsf{F2}}$ -bundles
 - \mathscr{P}_C is a principal TD_n -bundle
 - $\hat{\mathbf{p}}$ is a projection induced by strict morphism $\hat{\phi} : \mathsf{TD}_n \to \mathsf{TB}_n^{\mathsf{F2}}$
 - \check{p} induced by $\check{\phi} = \hat{\phi} \circ \phi_{\mathsf{flip}}$, flip morphism $\phi_{\mathsf{flip}} : \mathsf{TD}_n \to \mathsf{TD}_n$
- Differential refinement: (i.e. *B*-field+metric)
 - TB_n^{F2} does not come with adjustment, but
 - TD_n comes with very natural adjustment map
 - Have topological and full connection data on \mathscr{P}_C
 - $\circ\,$ Can reconstruct gerbe and bundle data on $\check{\mathscr{P}}$ and $\hat{\mathscr{P}}$
- (Also: generalization to affine tori.)

Example: Nilmanifolds

Geometry of string background $\check{\mathscr{G}}_{\ell} \to N_k$:

- Principal circle bundle over T^2 with $c_1 = k$
- Subordinate to $\mathbb{R}^2 \to T^2$ and with $\mathsf{U}(1) \cong \mathbb{R}/\mathbb{Z}$

 $(x,y,z)\sim (x,y+1,z)\sim (x,y,z+1)\sim (x+1,y,z-ky)$

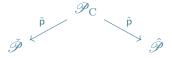
- Local connection form: $A(x,y) = kx \, dy \in \Omega^1(\mathbb{R}^2)$
- Kaluza-Klein metric: $g(x, y, z) = dx^2 + dy^2 + (dz + kx dy)^2$

T-duality:

$$(\check{\mathscr{G}}_{\ell} \to N_k) \iff (\hat{\mathscr{G}}_k \to N_{\ell})$$

Have: full interpretation in terms of higher bundle Kim, CS (2022)

Example: Nilmanifolds with principal 2-bundles



Lie 2-group:

$$\mathsf{TD}_1 \ \coloneqq \ \left(\mathbb{Z}^2 \times \mathsf{U}(1) \stackrel{\mathsf{t}}{\longrightarrow} \mathbb{R}^2\right)$$

Topological cocycle data:

$$\begin{split} g &= \begin{pmatrix} \hat{g}, \ \hat{\xi} \\ \check{g}, \ \check{\xi} \end{pmatrix} , \begin{array}{l} \hat{g}(x, y; x', y') = \mathbb{1} , & \hat{\xi}(x, y; x', y') = \ell(x' - x)y , \\ \check{g}(x, y; x', y') = \mathbb{1} , & \check{\xi}(x, y; x', y') = k(x' - x)y , \\ m &= \begin{pmatrix} \hat{m} \\ \check{m} \end{pmatrix} , \begin{array}{l} \hat{m}(x, y; x', y'; x'', y'') = -\ell(x'' - x')(y' - y) , \\ \check{m}(x, y; x', y'; x'', y'') = -k(x'' - x')(y' - y) , \\ \phi &= \frac{1}{2}k\ell \left(y'(xx'' - xx' - x'x'') - (x'' - x')(y'^2 - y^2)x \right) \end{split}$$

Cocycle data of differential refinement:

$$\begin{split} A &= \begin{pmatrix} \check{A} \\ \hat{A} \end{pmatrix} = \begin{pmatrix} kx \, \mathrm{d}y \\ \ell x \, \mathrm{d}y \end{pmatrix} , \quad B = 0 , \quad \Lambda = \frac{1}{2} k \ell (xx' \, \mathrm{d}y + (xy + x'y' + y^2(x' - x)) \, \mathrm{d}x) \\ \text{Can reconstruct both string backgrounds fully.} \end{split}$$

The T-duality group from Kaluza-Klein Reduction

The group TD_n from Kaluza–Klein reduction

Observation:

T-duality is intimately linked to Kaluza-Klein reduction:

- Gysin sequence contains fiber integration
- Metric on total space given by Kaluza–Klein metric
- Literature: e.g. Berman (2019), Alfonsi (2019), ...
- $\bullet\,$ Geometric objects from maps into classifying spaces $\mathcal{C}.$
- Note: currying $C^0(X \times T^n, \mathcal{C}) \cong C^0(X, C^0(T^n, \mathcal{C}))$
- Non-trivial fibrations: cyclic torus space: $C^0(T^n, \mathcal{C}) / / \mathrm{U}(1)^n$

Example TD_1 from KK-reduction of gerbe on circle bundle

- Gerbe: $C^0(P, \mathcal{C})$ with $\mathcal{C} = \mathsf{BBU}(1)$
- Cyclic loop space: $LBBU(1)//U(1) \cong B(BU(1) \times U(1) \times U(1))$
- Replace U(1) with $\mathbb{Z} \to \mathbb{R}$: $\mathsf{TD}_1 := (U(1) \times \mathbb{Z}^2 \xrightarrow{\mathsf{t}} \mathbb{R}^2)$
- Iterate: TD_n

Automorphisms of TD_n

Abstract nonsense:

- Natural definition of morphism of 2-groups
- Automorphisms of 2-group form naturally a 2-group
- 2-group action $\mathscr{G} \curvearrowright \mathscr{H}$: morphism $\mathscr{G} \to \operatorname{Aut}(\mathscr{H})$

Automorphisms of the 2-group TD_n :

- Restrict to "reasonable" automorphisms
- These are parameterized by $\mathsf{GO}(n,n;\mathbb{Z}) imes \mathsf{Sym}(2n;\mathbb{Z})$
- Recover T-duality group for affine torus bundles
- Neither this group nor $\mathsf{GO}(n,n;\mathbb{Z})$ fully acts on TD_n
- What works: weak (unital) Lie 2-group

 $\mathscr{GO}(n,n;\mathbb{Z}) \coloneqq \left(\begin{array}{c} \mathsf{GO}(n,n;\mathbb{Z}) \times \mathbb{Z}^{2n} \Longrightarrow \mathsf{GO}(n,n;\mathbb{Z}) \end{array} \right)$

- Explicit: geometric subgroup, B- and β -trafos, T-dualities
- \Rightarrow arrange everything based on $\mathscr{GO}(n,n;\mathbb{Z})$

Groupoid bundles for T-folds

T-folds from groupoid bundles

Recall our construction of TD_1 :

- Gerbe: $C^0(P, C)$ with $C = \mathsf{BBU}(1)$
- Cyclic loop space: $LBBU(1)//U(1) \cong B(BU(1) \times U(1) \times U(1))$
- Replace U(1) with $\mathbb{R} \times \mathbb{Z} \rightrightarrows \mathbb{R}$: $\mathsf{TD}_1 := (\mathbb{Z}^2 \times \mathsf{U}(1) \stackrel{\mathsf{t}}{\longrightarrow} \mathbb{R}^2)$

• Iterate: TD_n

Last point was sloppy, one obtains no 2-group, but 2-groupoid!

For T-folds: at least two T-duality directions \Rightarrow 2-groupoid!

Question: What is the appropriate groupoid here?

The 2-groupoid \mathscr{TD}_n

• Narain moduli space for affine circle bundles:

 $GM_n = \mathsf{GO}(n, n; \mathbb{Z}) \setminus \mathsf{O}(n, n; \mathbb{R}) / (\mathsf{O}(n; \mathbb{R}) \times \mathsf{O}(n; \mathbb{R}))$ =: $\mathsf{GO}(n, n; \mathbb{Z}) \setminus Q_n$.

• Resolve into action groupoid:

 $\mathsf{GO}(n,n;\mathbb{Z})\ltimes Q_n \ \Rightarrow \ Q_n \ .$

- Extend to $\mathscr{GO}(n,n;\mathbb{Z})$ -action $(\mathscr{GO}(n,n;\mathbb{Z}) \cong \operatorname{Aut}(\mathsf{TD}_n))$
- Place TD_n -fiber over every point in Q_n
- Include action of $\mathscr{GO}(n,n;\mathbb{Z})$ on TD_n
- The result is the 2-groupoid \mathscr{TD}_n

T-duality as \mathscr{TD}_n -bundles

Recall: functorial description of (higher) principal bundles:

- Manifold X
- Cover/surjective submersion $\sigma: Y \to X$
- Cech groupoid $\check{\mathscr{C}}(Y \to X) \coloneqq (Y \times_X Y \rightrightarrows Y)$
- Top. principal G-bundle: functor $\check{\mathscr{C}}(Y \to M) \to \mathsf{BG}$

For \mathscr{TD}_n -bundle:

- Replace (higher) group BG by Lie 2-groupoid \mathscr{TD}_n
- For ordinary groupoids: e.g. gauged sigma models

A non-geometric T-duality is simply a \mathscr{TD}_n -bundle.

T-duality as \mathscr{TD}_n -bundles

Remarks:

- The T-duality group $\mathscr{GO}(n,n;\mathbb{Z}) \supset \mathsf{GO}(n,n;\mathbb{Z})$ is gauged!
- Explicitly visible: $GO(n, n; \mathbb{Z})$ -gluing of local data
- Matches topological discussion in Nikolaus, Waldorf (2018)
- Differential refinement imposes restriction on top. cocycles
- This describes all T-dualities between pairs of T-folds
- Concrete conditions for "half-geometric" T-dualities
- Concrete cocycles of the T-fold in the nilmanifold example

To describe *Q*-spaces/T-folds: (can) use higher instead of noncommutative geometry.

Augmented groupoid bundles for R-spaces

What about R-spaces?

- \bullet T-folds/Q-spaces relatively harmless, as locally geometric
- *R*-spaces are not even locally geometric
- But perhaps higher description still works?

Note:

- One T-duality direction: *B*-field \rightarrow 2-, 1-forms \Rightarrow Lie 2-group TD_n-bundles with connection
- Two T-duality directions: *B*-field \rightarrow 2-, 1-, 0-forms \Rightarrow Lie 2-groupoid \mathscr{TD}_n -bundles with connection
- Three T-duality directions: *B*-field \rightarrow 2-, 1-, 0-, (-1)-forms \Rightarrow Augmented Lie 2-groupoid \mathscr{TD}_n^{aug} -bundles with connection

T-duality as \mathscr{TD}_n^{aug} -bundles

Construction of \mathscr{TD}_n^{aug} :

- Augmentation by suitable space of R-fluxes
- Determined by finite version of tensor hierarchy
- Finite embedding tensor $\mathbb{R}^{2n} \to \mathsf{GO}(n,n;\mathbb{Z}) \subset \mathscr{GO}(n,n;\mathbb{Z})$
- plus some standard consistency conditions
- Beyond this, augmentation fairly trivial

Remarks on T-duality with \mathscr{TD}_n^{aug} -bundles:

- Explicit examples, e.g. from nilmanifolds
- Yields consistency conditions between Q- and R-fluxes
- All previously discussed cases included
- All previously discussed also for affine U(1)-bundles

To describe *R*-spaces:

(can) use higher instead of nonassociative geometry.

Summary

What has been done:

- Top. T-duality can be described using principal 2-bundles
- Differential refinement requires adjusted curvatures
- Explicit description of geometric T-duality with nilmanifolds
- T-duality group is really a 2-group derived from KK-reduction
- Can extend to *Q*-spaces or T-folds
- Can extend to *R*-spaces

Future work:

- Link some mathematical results to physical expectations
- Link to pre-NQ-manifold pictures and similar
- Non-abelian T-duality?
- U-duality

Thank You!